Monotonie funkce

$\boxed{CV9}$

$f'(x) > 0 \quad \forall x \in I \Rightarrow f$ roste v I

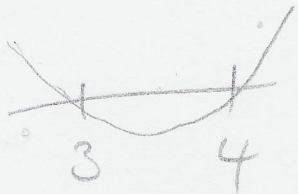
$f'(x) < 0 \quad \forall x \in I \Rightarrow f$ klesá v I

$f'(x_0) = 0 \ \ldots \ x_0 \in D_f$ je stacionární bod

$\boxed{Př}$ Určete intervaly monotonie, souřadnice lok extrémů

$$f(x) = \frac{x^3}{3} - \frac{7x^2}{2} + 12x + 5$$

$$f'(x) = x^2 - 7x + 12 = (x-3)(x-4)$$



$(-\infty; 3)$	$(3; 4)$	$(4; +\infty)$
$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
f roste	f klesá	f roste

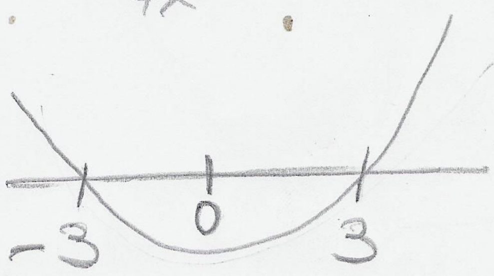
$\Downarrow$ $x_0 = 3$ je lok max

$\Downarrow$ $x_0 = 4$ je lok min

$\boxed{Př}$ $f(x) = \dfrac{x^2 - 10x + 9}{2x}$ $\quad D_f = \mathbb{R} \setminus \{0\}$

$$f'(x) = \frac{(2x-10)2x - (x^2 - 10x + 9)2}{4x^2} = \frac{4x^2 - \cancel{20x} - 2x^2 + \cancel{20x} - 18}{4x^2} =$$

$$= \frac{2(x^2 - 9)}{4x^2} = \frac{2(x-3)(x+3)}{4x^2}$$



$(-\infty;-3)$	$(-3;0)$	$(0;3)$	$(3;+\infty)$
$f'(x)>0$	$f'(x)<0$	$f'(x)<0$	$f'(x)>0$
f roste	f klesá	f klesá	f roste

$\Downarrow$ $x_0=-3$ je lokální maximum

$\Downarrow$ $x_0=3$ je lokální minimum

Př $f(x)=e^{\frac{1+x}{1-x}}$, $D_f=\mathbb{R}\setminus\{1\}$

$$f'(x)=e^{\frac{1+x}{1-x}}\,\frac{(1-x)-(1+x)(-1)}{(1-x)^2}=\underbrace{e^{\frac{1+x}{1-x}}}_{>0}\cdot\underbrace{\frac{2}{(1-x)^2}}_{>0}$$

$(-\infty;0)$	$(0;+\infty)$
$f'(x)>0$	$f'(x)>0$
f roste	f roste

Př $f(x)=e^{2x-x^2}$, $D_f=\mathbb{R}$

$$f'(x)=e^{2x-x^2}(2-2x)=\underbrace{2e^{2x-x^2}}_{>0}(1-x)$$

$(-\infty;1)$	$(1;+\infty)$
$f'(x)>0$	$f'(x)<0$
f roste	f klesá

$\Downarrow$ $x_0=1$ je lok maximum

Asymptoty v $\pm\infty$

1) $\lim\limits_{x \to +\infty} f(x) = \text{KONSTANTA}$ ... $f(x)$ má v $+\infty$ asymptotu $y = \text{KONSTANTA}$

2) $\lim\limits_{x \to +\infty} f(x) = \pm\infty$

a) $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = \pm\infty$ ... asymptota v $+\infty$ NEEXISTUJE

b) $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = \underset{\text{(KONSTANTA)}}{a}$ asymptota v $+\infty$

$\lim\limits_{x \to +\infty} [f(x) - ax] = b$ $\quad y = ax + b$

Asymptoty v krajním bodě $D_f \ni x_0$

$\lim\limits_{x \to x_0 \pm} f(x) = \pm\infty \Rightarrow$ $f(x)$ má v x_0 svislou asymptotu $x = x_0$

$\boxed{\text{Př}}$ $f(x) = \frac{2x-3}{2x}$, $D_f = \mathbb{R} \setminus \{0\}$

$\lim\limits_{x \to +\infty} \frac{2x-3}{2x} = 1$ ... $f(x)$ má v $+\infty$ asymptotu $y = \frac{1}{2}$

$\lim\limits_{x \to -\infty} \frac{2x-3}{2x} = 1$ ... $f(x)$ má v $-\infty$ asymptotu $y = \frac{1}{2}$

$\lim\limits_{x \to 0+} \frac{2x-3}{2x} = -\infty$, $\lim\limits_{x \to 0-} \frac{2x-3}{2x} = +\infty$ ... $f(x)$ má v 0 asymptotu $x = 0$

$\boxed{\text{Př}}$ $f(x) = \dfrac{x^2-10x+9}{2x}$ $D_f = R \setminus \{0\}$

$$\lim_{x\to+\infty} \frac{\frac{x^2-10x+9}{2x}}{x} = \lim_{x\to+\infty} \frac{x^2-10x+9}{2x^2} = \frac{1}{2} = a$$

$$\lim_{x\to+\infty} f(x) - ax = \lim_{x\to+\infty} \frac{x^2-10x+9}{2x} - \frac{1}{2}x =$$

$$= \lim_{x\to+\infty} \frac{x^2-10x+9-x^2}{2x} = \lim_{x\to+\infty} \frac{-10x+9}{2x} = -5$$

$f(x)$ má v $+\infty$ asymptotu $y = \frac{1}{2}x - 5$

$$\lim_{x\to-\infty} \frac{x^2-10x+9}{2x^2} = \frac{1}{2} = a$$

$$\lim_{x\to-\infty} \frac{x^2-10x+9}{2x} - \frac{1}{2}x = -5$$

$f(x)$ má v $+\infty$ asymptotu $y = \frac{1}{2}x - 5$

$$\lim_{x\to0_+} \frac{x^2-10x+9}{2x} = \frac{9}{0_+} = +\infty$$

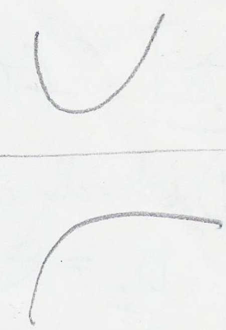
$$\lim_{x\to0_-} \frac{x^2-10x+9}{2x} = \frac{9}{0_-} = -\infty$$

$f(x)$ má v 0 asymptotu $x = 0$

Konvexnost / Konkávnost

$f''(x) > 0 \quad \cancel{\forall x \in I} \Rightarrow f$ je konvexní na I

$f''(x) < 0 \quad \forall x \in I \Rightarrow f$ je konkávní na I



inflexní bod: mění se konvexita konkávnost

Př $f(x) = \dfrac{x^2 - x - 2}{x - 3}$; $D_f = \mathbb{R} \setminus \{3\}$

$$f'(x) = \frac{(2x-1)(x-3) - (x^2 - x - 2)}{(x-3)^2} = \frac{2x^2 - 6x - x + 3 - x^2 + x + 2}{(x-3)^2} =$$

$$= \frac{x^2 - 6x + 5}{(x-3)^2}$$

$$f''(x) = \frac{(2x-6)(x-3)^{\cancel{2}} - (x^2 - 6x + 5)(2\cancel{(x-3)})}{(x-3)^{\cancel{4}}} =$$

$$= \frac{\cancel{2x^2} - 6x - 6x + 18 - \cancel{2x^2} + \cancel{12x} - 10}{(x-3)^3} = \frac{8}{(x-3)^3} =$$

$$= \frac{8}{(x-3)(x-3)(x-3)}$$

$- - - \mid + + +$

$\ominus \quad 3 \quad \oplus$

$(-\infty; 3)$	$(3; +\infty)$
$f''(x) < 0$	$f''(x) > 0$
konkávní	konvexní
	$3 \notin D_f \Rightarrow$ není inflexní bod

Průběh funkce

Př $f(x) = \frac{x^2+6x}{2-x}$

1) $D_f = R \setminus \{2\}$

2) limity v krajních bodech D_f

$\lim_{x \to +\infty} \frac{x^2+6x}{2-x} = \lim_{x\to+\infty} \frac{x^2(1+\frac{6}{x})}{x(\frac{2}{x}-1)} = \lim_{x\to+\infty} x \quad \lim_{x\to+\infty}\left(\frac{1+\frac{6}{x}}{-1+\frac{2}{x}}\right) =$

$= +\infty \cdot (-1) = \boxed{-\infty}$

$\lim_{x \to -\infty} \frac{x^2+6x}{2-x} = \dots = \lim_{x\to-\infty} x \cdot \lim_{x\to-\infty}\left(\frac{1+\frac{6}{x}}{-1+\frac{2}{x}}\right) = -\infty \cdot (-1) \boxed{+\infty}$

$\lim_{x \to 2^+} \frac{x^2+6x}{2-x} = \frac{16}{0-} = \boxed{-\infty}$

$\lim_{x \to 2^-} \frac{x^2+6x}{2-x} = \frac{16}{0+} = \boxed{+\infty}$

$\Bigg\}$ f(x) má asymptotu v 2
$x = 2$

3) průsečík s osami

$P_y = [0;0]$, $P_{x_1} = [0,0]$, $P_{x_2} = [-6;0]$

$x^2+6x=0$

$x(x+6)$

$x_1 = 0$

$x_2 = -6$

4) asymptoty

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\cancel{x^2}\left(1+\frac{6}{x}\right)}{\cancel{x^2}\left(\frac{2}{x}-1\right)}=-1=a$$

$$\lim_{x\to+\infty} f(x)-ax=\lim_{x\to+\infty}\frac{x^2+6x}{2-x}+x=$$

$$=\lim_{x\to+\infty}\frac{\cancel{x^2}+6x+2x-\cancel{x^2}}{2-x}=\lim_{x\to+\infty}\frac{8x}{2-x}=-8=b$$

f(x) má asymptotu v $+\infty$: $\boxed{y=-x-8}$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=-1=a$$

$$\lim_{x\to-\infty} f(x)-ax=-8=b$$

f(x) má asymptotu v $-\infty$ je $y=-x-8$

5) první derivace

$$f'(x)=\frac{(2x+6)(2-x)-(x^2+6x)(-1)}{(2-x)^2}=\frac{4x-2x^2+12-6x+x^2+6x}{(2-x)^2}=$$

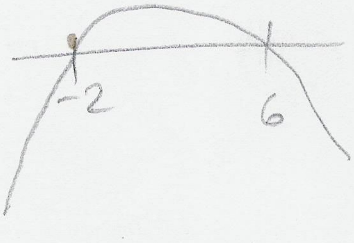
$$=\frac{-x^2+4x+12}{(2-x)^2}=\frac{-(x^2-4x-12)}{(2-x)^2}=\frac{-(x-6)(x+2)}{(2-x)^2}$$

6) monotonie

$(-\infty;-2)$	$(-2;2)$	$(2;6)$	$(6;+\infty)$
$f'(x)<0$	$f'(x)>0$	$f'(x)>0$	$f'(x)<0$
f klesá	f roste	f roste	f klesá
$\Downarrow$			$\Downarrow$

$x_0=-2$ je lok min

$x_0=6$ je lok max



7) extrémy $\leftarrow$

8) druhá derivace

$$f''(x) = \frac{(-2x+4)(2-x)^{\cancel{2}\,1} - (-x^2+4x+12)\,2\cancel{(2-x)}(-1)}{(2-x)^{\cancel{4}\,3}} =$$

$$= \frac{-\cancel{4x} + \cancel{2x^2} + 8 - \cancel{4x} - \cancel{2x^2} + \cancel{8x} + 24}{(2-x)^3} =$$

$$= \frac{32}{(2-x)^3} = \frac{32}{(2-x)(2-x)(2-x)}$$

9) konvexita/konkávnost

$$\begin{array}{ccc} + + + & | & - - - \\ (+) & 2 & (-) \end{array}$$

$(-\infty; 2)$	$(2; +\infty)$
$f''(x) > 0$	$f''(x) < 0$
konvexní	konkávní

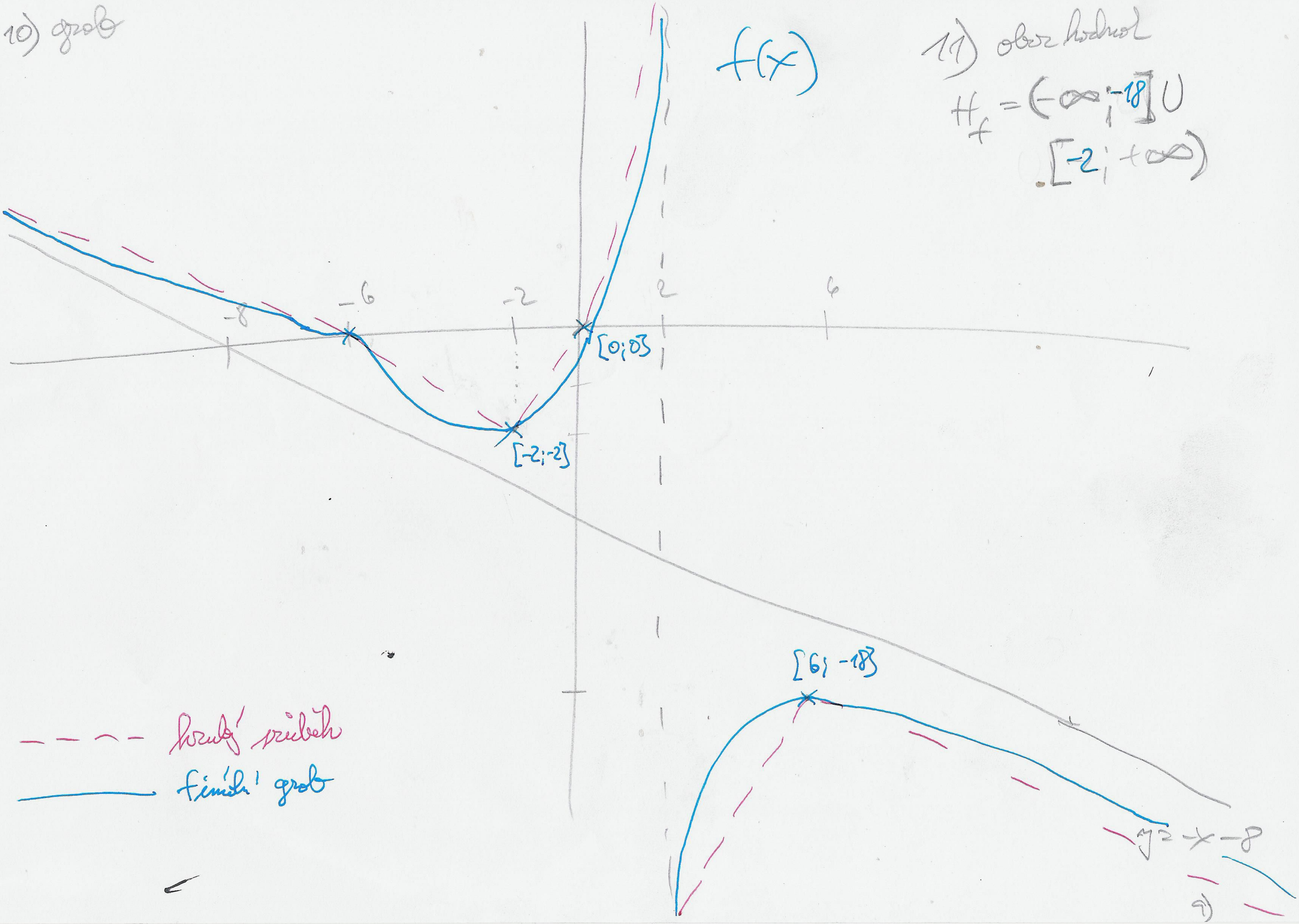
$x_0 = 2$ není inflexní bod $\notin D_f$

od) 7 dořešit extrémy

lok min $x_0 = -2$ $\rightarrow$ $f(-2) = \frac{-8}{4} = -2$

lok max $x_0 = 6$ $\quad f(6) = \frac{36+36}{2-6} = -9 - 9 = -18$

10) graf



11) obor hodnot

$H_f = (-\infty; -18] \cup [-2; +\infty)$